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Scalar Radiation from a Point Source

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Abstract

We consider a classical scalar field, obeying the inhomogeneous Klein–Gordon equation, in the case of a single point source. We propose a definition for the radiated energy-momentum and give an expression for it in terms of the prescribed world line of the source where we assume that the acceleration vanishes outside a finite interval. We find that only the part of the world line with non-vanishing acceleration contributes to the radiation, which travels at all speeds less than but not equal to the speed of light. We briefly discuss the case with more than one point source.

1. *Introduction*

The classical electromagnetic field, which is normally considered to be the high photon density limit of the quantized field (Thirring, 1958), serves as a useful approximation in a wide variety of physical applications. In the same sense, we study the classical scalar field as a possible approximation to a meson field (Iwanenko & Sokolow, 1953).

The field φ obeys the Klein-Gordon equation

$$(\Box + m^2)\,\varphi(x) = \rho(x) \tag{1.1}$$

where m is a parameter of dimension L^{-1}, and $\Box$ is the d'Alembertian operator,

$$\Box = \partial_\mu\,\partial_\mu \tag{1.2}$$

The derivative with respect to x^μ,

$$\partial_\mu = \partial/\partial x^\mu \tag{1.3}$$

can also be represented by a subindex μ following a comma; Greek indices range from 0 to 3, Latin indices from 1 to 3. We use the time-favoring metric $g_{\mu\nu}$, i.e.,

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \tag{1.4}$$

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and the modified summation convention for repeated Greek sub-indices

$$a.b = a_\mu b_\mu = a_0 b_0 - \mathbf{a}.\mathbf{b} \tag{1.5}$$

We restrict our main considerations to the case of a single point source with a prescribed world line given by the parametric equations

$$\xi_\mu = \xi_\mu(\tau) \tag{1.6}$$

In analogy with the electromagnetic current density (Rohrlich, 1965)

$$j_\mu(x) = q \int_{-\infty}^{\infty} \delta(x - \xi(\tau))\, u_\mu(\tau)\, d\tau \tag{1.7}$$

where q is the charge of the particle, $u = d\xi/d\tau$ and $u^2 = 1$ (we use the proper time to parametrize the world line and set the speed of light $c = 1$), we define the scalar source

$$\rho(x) = g \int_{-\infty}^{\infty} \delta(x - \xi(\tau))\, d\tau \tag{1.8}$$

where g is the 'scalar charge' of the source. We expect the particle to radiate if it is accelerated, and our purpose is to find a way to determine the radiation. We study in detail the case in which the acceleration vanishes outside a finite interval, and we assume that the acceleration is sufficiently well-behaved that the various mathematical operations to be performed be valid.

Due to the fact that the field does not propagate with the speed of light, it is necessary to give a careful definition of what is meant by radiation, which is done in Section 2. In Section 3 we determine the fields at large distances from the source, which we then use in Section 4 to find an expression for the energy-momentum of the radiation field. Section 5 is devoted to a brief discussion of simple generalizations of the problem, and we make some concluding remarks in Section 6.

2. *Definition of Radiation*

To find the solution to the Klein-Gordon equation (1.1), we use the retarded Green's function (Bogoliubov & Shirkov, 1959; Morse & Feshbach, 1953; Iwanenko & Sokolow, 1953)

$$\Delta_R(x) = \frac{-1}{(2\pi)^4} \int \frac{d^4 k \exp(-ik.x)}{k^2 - m^2 + i\epsilon k_0/|k_0|} \tag{2.1}$$

$$= \frac{\theta(t)}{2\pi} \left\{ \delta(\lambda) - \frac{m}{2\lambda^{1/2}} \theta(\lambda) J_1(m\lambda^{1/2}) \right\} \tag{2.2}$$

where $\theta(t)$ is the step function, and $\lambda = x^2$. The general solution of equation (1.1) then is

$$\varphi(x) = \varphi^{\text{in}}(x) + \int \rho(x')\, \Delta_R(x - x')\, d^4 x' \tag{2.3}$$

where $\varphi^{\text{in}}(x)$ is the solution of the homogeneous Klein-Gordon equation that satisfies the appropriate initial conditions given for $\varphi(x)$ in the remote past.

A simple example, that also serves as a check on the source (1.8), is the determination of the field of a particle in uniform motion. We have $\varphi^{\text{in}} = 0$, and

$$\xi_\mu(\tau) = \delta_{\mu 0}\, \tau \tag{2.4}$$

in the rest frame of the particle. We use Δ_R given by equation (2.1) to obtain the Yukawa field

$$\varphi(x) = \frac{g \exp\,[-mr(x)]}{4\pi r(x)} \tag{2.5}$$

where $r(x) = |\mathbf{x}|$. In an arbitrary Lorentz frame, $r(x)$ is the distance from the field point x to the world line of the particle.

To find the field of a particle with arbitrary prescribed world line and no incoming field, we substitute the source (1.8) and the Green's function (2.2) into equation (2.3) to obtain

$$\varphi(x) = g \int_{-\infty}^{\infty} d\tau \Delta_R(x - \xi(\tau)) \tag{2.6}$$

$$= \frac{g}{4\pi (x - \xi_R)\,.\,u_R} - \frac{mg}{4\pi} \int_{-\infty}^{\tau_R} \frac{J_1(m\lambda_\tau^{1/2})}{\lambda_\tau^{1/2}}\, d\tau \tag{2.7}$$

where $\lambda_\tau = (x - \xi(\tau))^2$, $\xi_R = \xi(\tau_R)$, $u_R = u(\tau_R)$, and $\tau_R(x)$ is the proper time of the intersection of the backward light cone from x and the world line of the particle, i.e., the solution of $(x - \xi(\tau_R))^2 = 0$ with $\xi_0(\tau_R) < t$. We change the integration variable to

$$\zeta = m\lambda_\tau^{1/2} \tag{2.8}$$

and the expression of the field becomes

$$\varphi(x) = \frac{g}{4\pi (x - \xi_R)\,.\,u_R} - \frac{g}{4\pi} \int_0^{\infty} d\zeta \frac{J_1(\zeta)}{(x - \xi(\tau))\,.\,u(\tau)} \tag{2.9}$$

where now $\tau = \tau(x, \zeta)$, and

$$\partial\tau(x, \zeta)/\partial\zeta = -\zeta[m^2(x - \xi)\,.\,u]^{-1} \tag{2.10}$$

Integration by parts gives

$$\varphi(x) = -g(4\pi m^2)^{-1} \int_0^\infty d\zeta \zeta J_0(\zeta)[-1 + (x-\xi) . w][(x-\xi) . u]^{-3} \quad (2.11)$$

$$= -(g/4\pi) \int_{-\infty}^{\tau_R} d\tau J_0(m\lambda_\tau^{1/2})[-1 + (x-\xi) . w][(x-\xi) . u]^{-2} \quad (2.12)$$

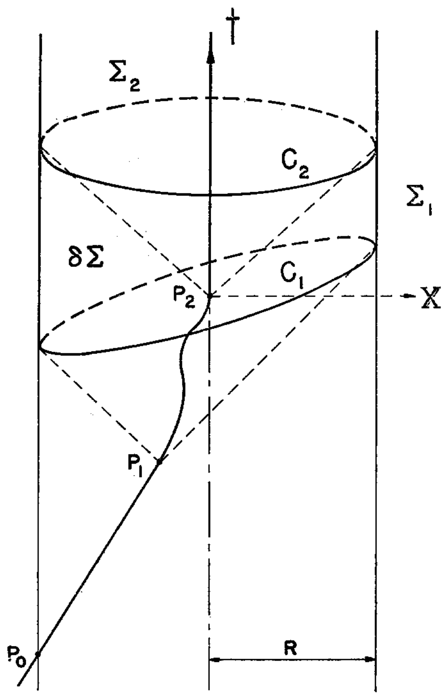

Figure 1.—World line of a particle accelerated over a finite interval P_1P_2. The radiation field is received over the 'cylindrical' surface Σ_1, which is separated into a 'surf' region $\delta\Sigma$ bounded by the curves C_1 and C_2 determined by the light cones from P_1 and P_2 and the 'wake' region Σ_2 above C_2.

where $w = du/d\tau$ is the four-acceleration of the particle. We see, from the above expressions for $\varphi(x)$, that the field at a certain point x is not just determined by the characteristics of the world line at the retarded event $\xi_R(x)$, but rather depends on the entire world line prior to τ_R. Consequently, there is no direct correspondence between a radiation field at an observation point x and a single event on the world line at which it can be said that the radiation was emitted.

Conversely, a given event on the world line contributes to the field at all future observation points. The cancellation of the term coming from the $\delta(\lambda)$ in the Green's function as a result of the integration by parts suggests that the radiation does not travel with the speed of light.

We now assume that the acceleration vanishes except for a finite interval $\tau_1 < \tau < \tau_2$. We choose a reference frame with origin at $\xi(\tau_2)$ and time axis along the outgoing world line, as shown in Fig. 1. In three-dimensional language, the particle comes in with constant

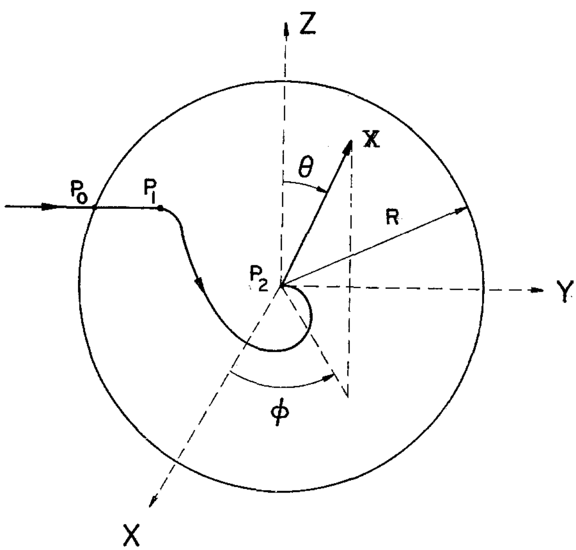

Figure 2.—Three-space trajectory of the particle. It is accelerated between P_1 and P_2, and it finally comes to rest in this particular Lorentz frame.

velocity, is accelerated over a finite time interval and comes to rest at a time we call $t = 0$ and at a point we choose as origin of our coordinate system, as shown in Fig. 2. We construct a fixed three-dimensional sphere of large radius R centered about the origin and enclosing the region where the accelerated motion of the particle takes place; this sphere is represented by the 'cylinder' in Fig. 1 in Minkowski space. The intersection of this cylinder and the forward light cone from the event $\xi(\tau_1)$ is represented by the curve C_1. We note that the field falls off exponentially away from the incoming world line and is completely negligible on the cylinder below C_1, except for a local disturbance in the neighbourhood of the event $\xi(\tau_0)$, when the particle enters the sphere. We define the energy and momentum of the radiation

$$P_\nu^{\mathrm{rad}} = \lim_{R\to\infty} \int_{\Sigma_1} T_{\mu\nu}(x)\, d\sigma_\mu \tag{2.13}$$

where Σ_1 is the surface of the cylinder above C_1. We have intentionally excluded any contribution from the particle crossing the surface of the sphere. The stress-energy tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \mathscr{L} g_{\mu\nu} \tag{2.14}$$

where the Lagrangian density $\mathscr{L}$ is

$$\mathscr{L} = \tfrac{1}{2}(\varphi_{,\alpha}\,\varphi_{,\alpha} - m^2\varphi^2) \tag{2.15}$$

3. *Faraway Fields*

In order to compute P_ν^{rad}, we have to determine the behavior of $\varphi(x)$ and $\varphi_{,\mu}(x)$ on Σ_1 for large values of R. It is convenient to break up Σ_1 into two parts, $\delta\Sigma$ and Σ_2, separated by the intersection C_2 of the forward light cone from $\xi(\tau_2)$ and the cylinder, as shown in Fig. 1.

We first study $\varphi(x)$ for a point x on $\delta\Sigma$. The integral in Eq. (2.11) can be broken up into two pieces, one from 0 to $\zeta_1(x)$, in which $w \neq 0$, and the other from ζ_1 to ∞, in which $w = 0$; the value ζ_1 is

$$\zeta_1(x) = m[(x - \xi(\tau_1))^2]^{1/2} \tag{3.1}$$

Because t on $\delta\Sigma$ is bounded by

$$R \geqslant t \geqslant R - [|\xi_0(\tau_1)| + |\boldsymbol{\xi}(\tau_1)|] \tag{3.2}$$

where

$$|\xi_0(\tau_1)| + |\boldsymbol{\xi}(\tau_1)| = O(R^0) \tag{3.3}$$

we have

$$\zeta_1(x) = O(R^{1/2}) \tag{3.4}$$

when x is a finite distance away from C_1. In this case, we break up the integral from 0 to ζ_1 further into a part from 0 to $\zeta' = O(R^{(1/2)-\varepsilon})$, where ε is an arbitrary positive number less than $\frac{1}{2}$, and one from ζ' to ζ_1. Equation (2.10) shows that

$$\tau(\zeta') - \tau_R = O(R^{-2\varepsilon}) \tag{3.5}$$

since

$$(x - \xi)\,.\,u = O(R) \tag{3.6}$$

so that

$$\int_0^{\zeta'} d\zeta\zeta J_0(\zeta)[-1+(x-\xi).w][(x-\xi).u]^{-3}$$

$$\approx [-1+(x-\xi_R).w_R][(x-\xi_R).u_R]^{-3}\int_0^{\zeta'} d\zeta\zeta J_0(\zeta) \quad (3.7)$$

since the quantities in brackets are essentially constant over this range of integration. We evaluate

$$\int_0^{\zeta'} d\zeta\zeta J_0(\zeta) = \zeta' J_1(\zeta') = O(R^{(1/4)-(\varepsilon/2)}) \quad (3.8)$$

whence

$$\int_0^{\zeta'} d\zeta\zeta J_0(\zeta)[-1+(x-\xi).w][(x-\xi).u]^{-3} = O(R^{(-7/4)-(\varepsilon/2)}) \quad (3.9)$$

In the range $\zeta' \leqslant \zeta \leqslant \zeta_1$, we can use the asymptotic form of the Bessel function. We have, using equation (2.12),

$$\int_{\tau_1}^{\tau'} d\tau J_0(m\lambda_\tau^{1/2})[-1+(x-\xi).w][(x-\xi).u]^{-2} = O(R^{-5/4}) \quad (3.10)$$

since the range of integration is finite. We have disregarded the fact that the Bessel function behaves like $R^{(-1/4)+(\varepsilon/2)}$, instead of $R^{-1/4}$, at the lower limit and its immediate vicinity, which has no influence on the overall behavior of the integral, as can also be seen when ζ is used as the variable of integration. Note that this is an estimate for the highest order in R it can have, and that the field can be strongly decreased due to destructive interference from the oscillating behavior of the Bessel function.† For the piece where $w = 0$, we note that

$$x-\xi = (x-\xi).uu - rn \quad (3.11)$$

where n is the unit normal from the point x to the incoming world line, and $r = O(R)$ is the corresponding distance. Squaring equation(3.11), we obtain

$$(x-\xi).u = (\zeta^2/m^2 + r^2)^{1/2} \quad (3.12)$$

† See the discussion after equation 4.7 below.

whence, using the asymptotic form of J_0 and integrating by parts, we have

$$\int_{\zeta_1}^{\infty} d\zeta\zeta J_0(\zeta)[(x-\xi)\,.\,u]^{-3}$$

$$\approx (2/\pi)^{1/2}\int_{\zeta_1}^{\infty} d\zeta\zeta^{1/2}\cos(\zeta-\tfrac{1}{4}\pi)(\zeta^2/m^2+r^2)^{-3/2}$$

$$=-(2/\pi)^{1/2}\zeta_1^{1/2}\sin(\zeta_1-\tfrac{1}{4}\pi)(\zeta_1{}^2/m^2+r^2)^{-3/2} \tag{3.13}$$

$$-(2/\pi)^{1/2}\int_{\zeta_1}^{\infty} d\zeta\sin(\zeta-\tfrac{1}{4}\pi)\,\tfrac{1}{2}\zeta^{-1/2}(\zeta^2/m^2+r^2)^{-5/2}(-5\zeta^2/m^2+r^2)$$

$$=O(R^{-5/2})$$

since the integrated term is of order $R^{-11/4}$ and for the integral we have

$$\left|\int_{\zeta_1}^{\infty} d\zeta\sin(\zeta-\tfrac{1}{4}\pi)\,\zeta^{-1/2}(\zeta^2/m^2+r^2)^{-5/2}(-5\zeta^2/m^2+r^2)\right|$$

$$\leqslant \int_{\zeta_1}^{\infty} d\zeta\zeta^{-1/2}(\zeta^2/m^2+r^2)^{-3/2}[(5\zeta^2/m^2+r^2)/(\zeta^2/m^2+r^2)] \tag{3.14}$$

$$\leqslant 5m^{1/2}r^{-5/2}\int_0^{\infty} dzz^{-1/2}(z^2+1)^{-3/2}=O(R^{-5/2})$$

where $z=\zeta/(mr)$.

If the point x is so close to the curve C_1 that $\zeta_1(x)=o(R^{1/2})$, we can set $\zeta'=\zeta_1$ and the integral in equation (3.10) does not appear, and $\varphi(x)$ goes to zero more rapidly as $R\to\infty$.

Now we turn to the behavior of $\varphi(x)$ for x on Σ_2. The integral in equation (2.11) can be broken up into three pieces, indicated symbolically by

$$\int_0^{\infty}=\int_0^{\zeta_2}+\int_{\zeta_2}^{\zeta_1}+\int_{\zeta_1}^{\infty} \tag{3.15}$$

To estimate the first integral, we note that (Erdélyi, 1953)

$$\int_0^{\infty} d\zeta\zeta J_0(\zeta)(\zeta^2/m^2+R^2)^{-3/2}=m^2\exp(-mR)/R \tag{3.16}$$

from which we have to subtract the integral from ζ_2 to ∞, which has the form of the one in equation (3.13). We note that ζ_2 (and ζ_1) can now be of order R^α, where $\alpha > \frac{1}{2}$, but this poses no difficulties. Hence

$$\int_0^{\zeta_2} d\zeta\zeta J_0(\zeta)(\zeta^2/m^2 + R^2)^{-3/2} = O(R^{-5/2}) \tag{3.17}$$

The third integral in equation (3.15) is again the integral in (3.13), except that now r is zero at the field point where the extension of the incoming world line intersects the cylinder. The region about this point can be seen to introduce no changes, since for $r = 0$ we have $\zeta_1 \approx R/|\mathbf{u}(\tau_1)| = O(R)$, and the integral in equation (3.14) is now bounded by

$$5m^3 \int_{\zeta_1}^{\infty} d\zeta\zeta^{-7/2} = O(R^{-5/2}) \tag{3.18}$$

For the second integral, ξ is finite and can be neglected in terms such as $(x-\xi).u$ and $(x-\xi).w$, but not in $(x-\xi)^2$! If $t = R + O(R^\alpha)$ with $0 \leqslant \alpha < 1$, we have $\zeta_{1,2} = O(R^{(1/2)+(\alpha/2)})$, and we use the asymptotic form of J_0 to find that

$$\begin{aligned}\int_{\zeta_2}^{\zeta_1} d\zeta\zeta J_0(\zeta)\, x\,.\,w(x\,.\,u)^{-3} &= m^2 \int_{\tau_1}^{\tau_2} d\tau J_0(m\lambda_\tau^{1/2})\, x\,.\,w(x\,.\,u)^{-2} \\ &= O(R^{(-5/4)-(\alpha/4)})\end{aligned} \tag{3.19}$$

If $t = R + O(R^\alpha)$ with $\alpha \geqslant 1$, we have $\zeta_{1,2} = O(R^\alpha)$ and

$$\int_{\zeta_2}^{\zeta_1} d\zeta\zeta J_0(\zeta)\, x\,.\,w(x\,.\,u)^{-3} = O(R^{-(3/2)\alpha}) \tag{3.20}$$

In the small region immediately above C_2 $(\alpha < 0)$, the integral can be seen to be at most of order $R^{-5/2}$ by equation (3.10).

Summarizing, we have shown that the leading contributions to the field come from the region of the world line where $w \neq 0$, and that

$$\varphi(x) = O(R^{-5/4}), \qquad x \text{ on } \delta\Sigma \tag{3.21}$$

$$\varphi(x) = \left.\begin{cases} O(R^{(-5/4)-(\alpha/4)}), & 0 \leqslant \alpha \leqslant 1 \\ O(R^{-(3/2)\alpha}), & \alpha \geqslant 1 \end{cases}\right\}, \qquad x \text{ on } \Sigma_2 \tag{3.22}$$

where the parameter α comes from the condition $t = R + O(R^\alpha)$ for the field point.

An observer on the sphere in Fig. 2 would notice that, at a time given by a point on C_1 (in Fig. 1), a field of order at most $R^{-5/4}$, but possibly smaller due to destructive interference, is quickly built up and persists for times corresponding to $\delta\Sigma$. This 'surf' is then followed by a long 'wake', at later times corresponding to Σ_2, with a field of decreasing intensity.

We now use equation (2.11) and

$$\partial_\mu \tau(x, \zeta) = (x_\mu - \xi_\mu)/(x - \xi) . u \tag{3.23}$$

to find

$$\varphi_{,\mu}(x) = -g(4\pi m^2)^{-1} \int_0^\infty d\zeta \zeta J_0(\zeta)\{[((x-\xi).u)^2]\, w_\mu - [3(-1+(x-\xi).w)(x-\xi).u]\, u_\mu + [(x-\xi).\dot{w}(x-\xi).u - 3(-1+(x-\xi).w)^2](x_\mu - \xi_\mu)\}[(x-\xi).u]^{-5} \tag{3.24}$$

where $\dot{w} = dw/d\tau$.

The behavior of $\varphi_{,\mu}(x)$ is the same as that of $\varphi(x)$ for large values of R.

We are now prepared to study the behavior of the energy and momentum of the field at large distances from the source.

4. *Radiated Energy-Momentum*

We break up the integral in equation (2.13) into two parts, one over the surf region $\delta\Sigma$ and the other over the wake region Σ_2. Equation (3.21) shows that there is no contribution from the surf region when $R \to \infty$, since the range of the integration over t remains finite.

A point $x_\mu = (t, \mathbf{x})$ on the wake region is defined by

$$t = R + O(R^\alpha) \tag{4.1}$$

$$\mathbf{x} = R\hat{R}(\Omega) \tag{4.2}$$

where $\hat{R}(\Omega)$ is the outward unit normal on the sphere in the direction given by the angles θ and φ. The surface element on the cylinder has to be taken pointing towards the time axis, since the cylinder is a timelike surface. We thus set $d\sigma_\mu = n_\mu d\sigma$, where

$$n_\mu = (0, -\hat{R}) \tag{4.3}$$

Then the energy is

$$P_0 = \lim_{R\to\infty} \int_R^\infty dt \int R^2\, d\Omega(-\varphi_{,0}\,\varphi_{,i})\, n_i \tag{4.4}$$

The $R^{-3/2}$ behavior of φ and $\varphi_{,\mu}$ when $\alpha = 1$ in equation (3.22) indicates that the limit in equation (4.4) is finite and nonzero, since the range of integration over t provides an additional power of R. We can also see from equation (3.22) that regions where $\alpha \neq 1$ do not contribute to the radiated energy. Since the speed of propagation of the field is of the order of $R/t = 1/[1 + O(R^{\alpha-1})]$, the physical significance of the preceding remark is that the radiation travels with speeds less than the speed of light.

We use the large R limit of $\varphi_{,\mu}$ in equation (3.24),

$$\varphi_{,\mu} \sim -(g/4\pi)\, x_\mu \int_{\tau_1}^{\tau_2} d\tau J_0(m\lambda_\tau^{1/2})[x\,.\,\dot{w}x\,.\,u - 3(x\,.\,w)^2](x\,.\,u)^{-4} \tag{4.5}$$

$$= -\frac{gm^2}{4\pi}\left[\frac{x_\mu}{x\,.\,u}\,\frac{J_1(m\lambda_\tau^{1/2})}{m\lambda_\tau^{1/2}}\right]_{\tau_1}^{\tau_2} + \frac{gm^4}{4\pi}x_\mu \int_{\tau_1}^{\tau_2} d\tau \frac{J_2(m\lambda_\tau^{1/2})}{m^2\lambda_\tau} \tag{4.6}$$

where we have integrated by parts twice, and we have set $w(\tau_1) = w(\tau_2) = 0$. Since the region where we are at a finite distance, that is, of order R^0, away from C_2 does not contribute to the radiation, we can start our integrals at $t = R + a$ instead of $t = R$ to be able to use the asymptotic form of the Bessel functions. Furthermore, we can neglect ξ in $\lambda_\tau = (x - \xi)^2$ when it is not in the argument of an oscillating function, in which case we set

$$\lambda_\tau^{1/2} \approx \lambda^{1/2} - \lambda^{-1/2} x\,.\,\xi \equiv s(x, \tau) \tag{4.7}$$

In the region of interest, where $t = R + O(R)$, we have $\lambda = O(R^2)$ and $x\,.\,\xi = O(R)$, so that, for a given x, s is equal to a (large) constant plus a slowly varying function of τ. On the other hand, when $t = R + O(R^0)$, $\lambda = O(R)$ only and the expansion in equation (4.7), as well as the neglect of the second term of the expansion for nonoscillating functions, are not justified, but we do not worry, since this region does not contribute to the radiation anyway. This is also a region where we get destructive interference from oscillating functions with arguments of the form $R^{1/2}f(\tau)$. We thus obtain from equation (4.6),

$$\varphi_{,\mu} \sim -\frac{g}{4\pi}\left(\frac{2m}{\pi}\right)^{1/2} \frac{x_\mu}{\lambda^{3/4}} \left\{ \left[\frac{\cos\,(ms - \frac{3}{4}\pi)}{x\,.\,u}\right]_{\tau_1}^{\tau_2} - \frac{m}{\lambda^{1/2}} \int_{\tau_1}^{\tau_2} d\tau \cos\,(ms - \tfrac{5}{4}\pi) \right\} \tag{4.8}$$

We substitute the expression (4.8) into (4.4), and we use (4.3) to find

$$P_0 = \lim_{R\to\infty} R^2 \int d\Omega \int_{R+a}^{\infty} dt\, Rt \frac{mg^2}{8\pi^3\lambda^{3/2}} \left\{ \left[\frac{\cos(ms - \frac{3}{4}\pi)}{x\,.\,u} \right]_{\tau_1}^{\tau_2} - \frac{m}{\lambda^{1/2}} \int_{\tau_1}^{\tau_2} d\tau \cos(ms - \tfrac{5}{4}\pi) \right\}^2 \tag{4.9}$$

where a is an arbitrary constant independent of R.

In order to compute the momentum $\mathbf{P}$, it is also necessary to have the asymptotic form of the field φ in equation (2.12)

$$\varphi \sim -(g/4\pi) \int_{\tau_1}^{\tau_2} d\tau J_0(m\lambda_\tau^{1/2})\, x\,.\,w(x\,.\,u)^{-2} \tag{4.10}$$

$$= \frac{g}{4\pi} \left[\frac{J_0(m\lambda_\tau^{1/2})}{x\,.\,u} \right]_{\tau_1}^{\tau_2} - \frac{gm^2}{4\pi} \int_{\tau_1}^{\tau_2} d\tau \frac{J_1(m\lambda_\tau^{1/2})}{m\lambda_\tau^{1/2}} \tag{4.11}$$

$$\approx \frac{g}{4\pi} \left(\frac{2}{\pi m} \right)^{1/2} \frac{1}{\lambda^{1/4}} \left\{ \left[\frac{\cos(ms - \frac{1}{4}\pi)}{x\,.\,u} \right]_{\tau_1}^{\tau_2} - \frac{m}{\lambda^{1/2}} \int_{\tau_1}^{\tau} d\tau \cos(ms - \tfrac{3}{4}\pi) \right\} \tag{4.12}$$

which is then substituted into

$$P_j = \lim_{R\to\infty} \int_R^{\infty} dt \int R^2 d\Omega [-\varphi_{,j}\varphi_{,i} - \tfrac{1}{2}\delta_{ij}(\varphi_{,\alpha}\varphi_{,\alpha} - m^2\varphi^2)]\, n_i \tag{4.13}$$

The angular distribution of the emitted radiation can be obtained from equations (4.4) and (4.13) by omitting the integrations over angles. It can be seen that $d\mathbf{P}/d\Omega$ is parallel to $\hat{R}$.

5. *Generalizations*

In Section 3 we showed that the leading contributions to φ and $\varphi_{,\mu}$ come from the part of the world line where $w \neq 0$ only. From equations (4.5) and (4.10) we see that the actual source of the radiation field is restricted to $\tau_1 < \tau < \tau_2$. It should then be possible to deform the surface of the cylinder to any shape as long as it encloses this segment of the world line and its spatial dimensions tend to infinity, in particular, to another cylinder whose axis is at an angle with the world line of the outgoing particle. The latter represents a three-dimensional spherical surface, with respect to which the particle is not finally at rest, but moves with constant velocity, and we have eliminated any contribution from the Yukawa field when the particle eventually passes out of the sphere.

The above considerations indicate how to calculate the radiation from a system of several particles that undergo accelerations in the same finite region of spacetime. One has merely to enclose this region

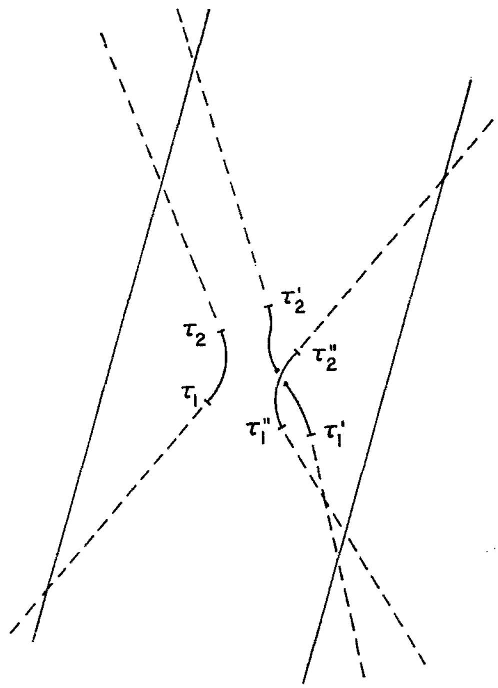

Figure 3.—World lines of several particles accelerated within a finite region of spacetime, included within the cylinder.

with a large cylinder and obtain the total field by superposition of the fields obtained from the segments. The axis of the cylinder can be any timelike direction, as shown in Fig. 3.

What we have discussed is an approximation to the case where the accelerations tend asymptotically to zero. It should be possible to extend the integrations in equations (4.5) and (4.10) over the whole world line if the accelerations go to zero sufficiently rapidly.

Since we have expressions for the radiation field and its derivatives, it is straightforward to compute other physical quantities, such as the radiated angular momentum.

6. *Concluding Remarks*

We have analyzed the problem of scalar radiation from a point source.

We have found that there is no contribution to the radiation field from those parts of the world line where the acceleration vanishes. The field at a large distance R from the acceleration region behaves initially (for times $t \lesssim R$) at most like $R^{-5/4}$. For much later times $[t = R + O(R)]$ we find a $R^{-3/2}$ behavior of the field, and this is the only region that contributes to the radiated energy momentum; the area of the large sphere provides a factor R^2 and the time integration, an additional power of R. We thus find radiation that travels at speeds $0 < v < 1$.

Although we have treated explicitly only the case of a single particle accelerated over a finite interval, generalizations to include more than one particle or particles with asymptotically vanishing accelerations appear straightforward.

References

Bogoliubov, N. N. and Shirkov, D. V. (1959). *Introduction to the Theory of Quantized Fields*, p. 151. Interscience Publishers, Inc., New York.

Erdélyi, A. (editor) (1953). *Higher Transcendental Functions*, Vol. 2, p. 95, equation (51). McGraw-Hill Book Company, Inc., New York.

Iwanenko, D. and Sokolow, A. (1953). *Klassische Feldtheorie*. Akademie-Verlag, Berlin.

Morse, P. M. and Feshbach, H. (1953). *Methods of Theoretical Physics*, Vol. 1, p. 856. McGraw-Hill Book Company, Inc., New York.

Rohrlich, F. (1965). *Classical Charged Particles*, p. 81. Addison-Wesley Publishing Company, Inc., Reading, Massachusetts.

Thirring, W. (1958). *Principles of Quantum Electrodynamics*, p. 12. Academic Press, Inc., New York.